EXERCISE – IV HINTS & SOLUTIONS

Sol.1 Logarithm function is continuous & differentiable in its domain.
for 0 < a < b ; f(x) is continious & differentiable

$f(a) = \log\left(\frac{a(a+b)}{a(a+b)}\right) = \log 1 = 0$ & $f(b) = 0$

Hence f(a) = f(b)
so rolle's theorem is applicable

Sol.2 Consider $f(x) = 3x^2 + px - 1$
f(x) is a polynomial function & hence continuous & differentiable in (–1, 1).
f(–1) & f(1) has opposite sign
hence at last one root b/w (–1, 1)

Sol.3 Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots\ldots a_{n-1}\,x$
f(x) has two roots 0 & α.
f(x) is continuous & differentiable, an f(x) is a polynomial so rolle's theorem is applicable :
$f'(x) = 0$ for some $c \in (0, \alpha)$.

Sol.4 f(x) = 1/x
f(x) is continious in $[-1, 0) \cup (0, 1]$
& also differentiable in $[-1, 1] - \{0\}$
so Lagrange's mean value theorem is not appeicable in [–1, 1]

Sol.5 Let $f(x) = x^3$

$f(x)$ is continuous on $[a, b]$ & differentiable on (a, b).

so using LMVT in (a, b)

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 = \frac{b^3 - a^3}{b - a}$$

$\Rightarrow 3c^2 = a^2 + ab + b^2$

Sol.6 Let $f(x) = x^n$

$f(x)$ is continuous & differentiable in $[a, b]$

so using LMVT

$$f'(c) = \frac{a^n - b^n}{a - b} \qquad \text{........(i)}$$

$a < c < b \Rightarrow a^{n-1} < c^{n-1} < b^{n-1}$

so $nb^{n-1}(a - b) < a^n - b^n < na^{n-1}(a - b)$ if $n > 1$

If $0 < n < 1$ then inequalities will be opposite sense

Sol.7 $f(x)$ is continious on $[0, 1]$ & differentiable on $(0, 1)$

$f(0) = f(1) = 0$

so using rolle's theorem

$f'(x) = 0$ for some $x \in [0, 1]$

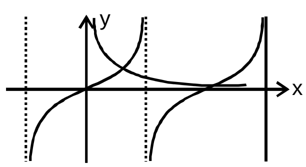
$f'(x) = \sin \pi x + x \text{ con } (\pi/x)(-\pi/x^2) = 0$

$\Rightarrow \tan \pi/x = \pi/x$

$x \in (0, 1)$; $x < 1 \Rightarrow 1/x > 1 \Rightarrow \pi/x > \pi$

$\Rightarrow \pi/x \in (\pi, \infty)$

so infinite solution



Sol.8 $f(x)$ is continuous & differentiable also $(x + 1)$ is continious in $(0, 5)$ & differentiable in $(0, 5)$.

so $g(x) = \dfrac{f(x)}{x - 1}$ is also continious

in $[0, 5]$ & differentiable in $(0, 5)$.

so there will be some $c \in (0, 5)$

i.e. $g'(c) = \dfrac{g(5) - g(0)}{5 - 0} \Rightarrow g'(c) = -\dfrac{5}{6}$

Sol.9 Let $h(x) = \dfrac{f(x)}{g(x)}$

Let α & β one two roots of $f(x)$.

so α & β one roots of $h(x)$ also.

$f(x)$ & $g(x)$ both ne continuous & differetiable so $h(x)$ is also.

& $h(\alpha) = h(\beta) = 0$

so rolle's theorem is applicable in $[\alpha, \beta]$

let $g(x) \neq 0$ in (α, β)

so $h'(x) = 0$ for $x \in (\alpha, \beta)$

$\Rightarrow g'(x).f(x) - f'(x).g(x) = 0$

but $g'(x).f(x) - f'(x)\, g(x) \neq 0$ (given)

so $g(x) = 0$ in (α, β)

so $g(x)$ has atleast one root b/w any two roots of $f(x)$.

Sol.10 If f is continuous in $[a, b]$

& differentiable in (a, b) then

there exist $x_0 \in (a, b)$

such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a} \qquad \text{.......(i)}$$

using $f(b) = \dfrac{b.f(a)}{a}$ in (i)

& let $a = x_0$

then $f'(x_0) = \dfrac{f(x_0)}{f(x_0)}$

Sol.11 $f(x) = (x - a)^m (x - b)^n$

As $f(x)$ is a polynomial, $f(x)$ is continuous in $[a, b]$

also $f(x)$ is differentiable in (a, b) & $f(a) = f(b)$

so rolle's theorem is applicable for $f(x)$

Sol.12 f is continuous in $[a, b]$ & differentiable in (a, b)

so there exist some c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{.........(i)}$$

also $f(b) - f(a) > 0$ & $b - a > 0$

so $f'(c) > 0$ for some $c \in (a, b)$

Sol.13 $f(x) = 4x^3 - 3x^2 - 2x + 1$

$f(x)$ is continious in $[0, 1]$ & differentiable in $(0, 1)$, as $f(x)$ is a polynomial also $f(0) = 1$ & $f(1) = 0$.

so there will be some $c \in (0, 1)$ i.e. $f(c) = 0$.

Sol.14 **(a)** Let $f(x) = \tan x - x$

$f'(x) = \sec^2 x - 1 > 0$ for $x \in (0, \pi/2)$

so f(x) is a increasing function

so $f(x) > f(0)$

hence $\tan x - x > 0 \Rightarrow \tan x > x$

(b) as in (a).

Sol.15 Let $f(x) = e^{\alpha x} . f(x)$

$f(a) = f(b) = 0$

& f(x) is continuous and differentiable in (a, b)

so rolle′s theorem is applicable

so $f'(x) = 0$ for some $x \in (a, b) \Rightarrow \alpha f(x) + f'(x) = 0$.

Sol.16 f(x) should be continious in [0, 2]

& differentiable in (0, 2).

so $f(0^+) = f(0) \Rightarrow a = 3$

$f(1^-) = f(1^+) \Rightarrow 5 = m + b$ ......(i)

$f'(1^-) = f'(1^+) \Rightarrow m = 1$ & $b = 4$

Sol.17 $f'(x) = \pm 5$ (for bonding function)

$\Rightarrow f(x) = \pm (5x + c)$

$f(-2) = 1 \Rightarrow f(x) = \pm (5x + 9)$.

Sol.18 Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x)$

h(x) is also continious & differentiable for all $x \ge 0$.

using LMVT ; $h'(x) = \dfrac{h(x) - h(0)}{x - 0}$ & $h(x) \le 0$

so $h(x) - h(0) \le 0 \Rightarrow h(x) \le h(0)$ $\because f'(x) - g'(x) \le 0$

$\Rightarrow f(x) - g(x) \le 0 \Rightarrow f(x) \le g(0)$ $\{\because f(0) = g(x)\}$

In all $x \ge 0$

Sol.19 f(x) is continious on [a, b] & differentiable on (a, b)

so using LMVT ;

$f'(C) = \dfrac{f(b) - f(a)}{b - a}$ .....(i)

for some C_1 & $C_2 \in (a, b)$ using (i)

$f'(C_1) + f'(C_2) = 2$

Sol.20 f(x) & f ′(x) are differentiable & continuous in [0, 1]

for f(x) there will be some 'c' $\in (0, 1)$ i.e. $f'(c) = 0$

Case–1 : x = c then

$f'(x) = f'(c) = 0 \Rightarrow [f'(x)] = |0| = 0 < 1$

Case–2 : x > c, by LMVT in [c, x]

$\dfrac{f'(x) - f'(c)}{x - c} = f''(\alpha)$ for $c < \alpha < x$

$f'(x) = (x - c) f''(\alpha)$ $\{\because f'(c) = 0\}$

$|f'(x)| = |(x - c)| . |f''(\alpha)|$

$x \in [0, 1]$ & $c \in (0, 1) \Rightarrow |x - c| < 1$ &

$f''(x) < 1 \ \forall \ x\ (0, 1)$.

so $|f'(x)| < 1 \ \forall \ x \in [0, 1]$.

Case–3 : If x < c then

$\dfrac{f'(c) - f'(x)}{c - x} = f''(\alpha) \Rightarrow |-f'(x)| = |c - x|\, |f''(\alpha)|$

$\Rightarrow |f'(x)| < 1$ so $|f'(x)| < 1$ for all $x \in [0, 1]$.

Sol.21 Let $h(x) = f(x) - 3g(x)$

h(x) is continious on [0, 1]

& differentiable on (0, 1).

$h(0) = 5$ & $h(1) = 5$

so rollis theorem is applicable for h(x) in [0, 1]

i.e. some $c \in (0, 1)$ such that $h(c) = 0$

$\Rightarrow f'(c) - 3g'(c) = 0 \Rightarrow f'(c) = 3\,g'(c)$

Sol.22 By LMVT

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$; $\phi'(c) = \dfrac{\phi(b) - \phi(a)}{b - a}$

$\psi'(c) = \dfrac{\psi(b) - \psi(a)}{b - a}$

By expanding the determinant

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix}$$

$\Rightarrow$ $f(a)\,[\phi(b)\,\psi'(c) - \phi'(c)\,\psi(b)]$

$- f(b)\,[\phi(a)\,\psi'(c) - \psi(a)\,\phi'(c)]$

$- f'(c)\,[\phi(a)\,\psi(b) - \psi(a)\,\phi(b)]$

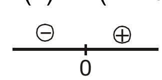
by puting the value of $f'(c)$, $\phi'(c)$ & $\psi'(c)$

and will get the value equal to zero

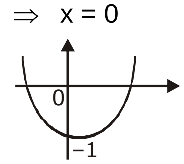
Sol.23 Let $f(x) = x^2 - x \sin x - \cos x$

$f'(x) = 2x - \sin x - x \cos x + \sin x$

$f'(x) = x(2 - \cos x) = 0 \Rightarrow x = 0$



$f(0) = -1$



Two real value of x possible where $f(x) = 0$

Sol.24 Using LMVT

$f'(x) = \dfrac{f(a) - f(-a)}{a - (-a)} = 1$ so $f'(x) = 1$

$\Rightarrow f(x) = x + c$

$c = 0$ so $f(x) = x$

hence ; $f(0) = 0$